

# On renormalization of Poisson–Lie T–plural sigma models

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## Abstract

Conditions for covariance of the one-loop renormalization group equations with respect to Poisson Lie T–plurality of sigma models are formulated and investigated. Examples of nonequivalence of these equations under the Poisson Lie T–plurality are given. Discrepancies with renormalization group equations on the Drinfel’d double are discussed. Role of ambiguities in renormalization group equations of Poisson–Lie sigma models with truncated matrices of parameters is presented.

Keywords: Sigma Models, String Duality, Renormalization Group

## 1 Introduction

One-loop renormalizability of Poisson–Lie dualizable  $\sigma$ –models and their renormalization group equations were derived in [1]. Covariance of renormalization group equations with respect to Poisson–Lie T–duality was proved in [2]. That suggests that also properties of quantum  $\sigma$ –models can be given by Drinfel’d doubles and not their decompositions into Manin triples. This was indeed claimed in [3] where a renormalization on the level of sigma models defined on Drinfel’d double was proposed. A natural way to independently verify this claim is to extend the proof of covariance of [2] to Poisson–Lie T–plurality.

Unfortunately, transformation properties of structure constants and matrix  $M$  (parameters of the models) under the Poisson–Lie T–plurality are much more complicated and it is difficult to see in general whether the renormalization group equations are covariant with respect to these transformations or not. That’s why we decided to check it first on the examples and in this paper we are going to show that the one–loop renormalization group equations, as derived in [1], are in general not covariant under the Poisson–Lie T–plurality.

An assumption in the renormalizability proof [1] is that there is no a priori restriction on elements of matrix  $M$  that together with the structure of the Manin triple determine the models. It was noted in [2] that the renormalization group equations need not be consistent with truncation of the parameter space. On the other hand there are some ambiguities in the renormalization group equations and we are going to show how they can be used in the choice of one–loop  $\beta$  functions for given truncation.

## 2 Review of Poisson–Lie T–plurality

For simplicity we shall consider  $\sigma$ –models without spectator fields, i.e. with target manifold isomorphic to a group. Let  $G$  be a Lie group and  $\mathcal{G}$  its Lie algebra. Sigma model on the group  $G$  is given by the classical action

$$S_E[g] = \int d^2x R_-(g)^a E_{ab}(g) R_+(g)^b, \quad (1)$$

where  $g : \mathbf{R}^2 \rightarrow G, (\sigma_+, \sigma_-) \mapsto g(\sigma_+, \sigma_-)$  and  $R_{\pm}$  are right-invariant fields  $R_{\pm}(g) := (\partial_{\pm} g g^{-1})^a T_a \in \mathcal{G}$  and  $E(g)$  is a certain bilinear form on the Lie algebra  $\mathcal{G}$ , to be specified below.

The  $\sigma$ –models that can be transformed by the Poisson–Lie T–duality are formulated (see [4, 5]) by virtue of Drinfel’d double  $D \equiv (G|\tilde{G})$  – a Lie group whose Lie algebra  $\mathcal{D}$  admits a decomposition  $\mathcal{D} = \mathcal{G} \dot{+} \tilde{\mathcal{G}}$  into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form  $\langle \cdot, \cdot \rangle$ . These decompositions are called Manin triples.

The matrices  $E(g)$  for such  $\sigma$ –models are of the form

$$E(g) = (M + \Pi(g))^{-1}, \quad \Pi(g) = b(g) \cdot a^{-1}(g) = -\Pi(g)^t, \quad (2)$$

where  $E_0$  is a constant matrix and  $a(g), b(g)$  are submatrices of the adjoint representation of the subgroup  $G$  on the Lie algebra  $\mathcal{D}$  defined as

$$gTg^{-1} \equiv Ad(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} \equiv Ad(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \quad (3)$$

where  $T_a$  and  $\tilde{T}^a$  are elements of dual bases of  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$ , i.e.

$$\langle T_a, T_b \rangle = 0, \quad \langle \tilde{T}^a, \tilde{T}^b \rangle = 0, \quad \langle T_a, \tilde{T}^b \rangle = \delta_a^b.$$

Origin of the Poisson–Lie T–plurality [4, 6] lies in the fact that in general several decompositions (Manin triples) of the Drinfel’d double may exist. Let  $\mathcal{D} = \hat{\mathcal{G}} \dot{+} \bar{\mathcal{G}}$  be another decomposition of the Lie algebra  $\mathcal{D}$  into maximal isotropic subalgebras. The dual bases of  $\mathcal{G}, \tilde{\mathcal{G}}$  and  $\hat{\mathcal{G}}, \bar{\mathcal{G}}$  are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} K & Q \\ R & S \end{pmatrix} \begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix}, \quad (4)$$

where the matrices  $K, Q, R, S$  are chosen in such a way that the structure of the Lie algebra  $\mathcal{D}$  in the basis  $(T_a, \tilde{T}^b)$

$$\begin{aligned} [T_a, T_b] &= f_{ab}^c T_c, \\ [\tilde{T}^a, \tilde{T}^b] &= \tilde{f}^{ab}_c \tilde{T}^c, \\ [\tilde{T}^a, T_b] &= f_{bc}^a \tilde{T}^c - \tilde{f}^{ac}_b T_c \end{aligned} \quad (5)$$

transforms to the similar one where  $T \rightarrow \hat{T}$ ,  $\tilde{T} \rightarrow \bar{T}$  and the structure constants  $f, \tilde{f}$  of  $\mathcal{G}$  and  $\tilde{\mathcal{G}}$  are replaced by the structure constants  $\hat{f}, \bar{f}$  of  $\hat{\mathcal{G}}$  and  $\bar{\mathcal{G}}$ . The duality of both bases requires

$$\begin{pmatrix} K & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} S^t & Q^t \\ R^t & K^t \end{pmatrix}. \quad (6)$$

The  $\sigma$ –model obtained by the Poisson–Lie T–plurality is defined analogously to (1,2) where

$$\hat{E}(\hat{g}) = (\hat{M} + \hat{\Pi}(\hat{g}))^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}(\hat{g}) \cdot \hat{a}^{-1}(\hat{g}) = -\hat{\Pi}(\hat{g})^t,$$

$$\hat{M} = (M \cdot Q + S)^{-1} \cdot (M \cdot K + R) = (K^t \cdot M - R^t) \cdot (S^t - Q^t \cdot M)^{-1}. \quad (7)$$

Classical solutions of the two  $\sigma$ –models are related by two possible decompositions of  $l \in D$ ,

$$l = g\tilde{h} = \hat{g}\bar{h}. \quad (8)$$

Examples of explicit solutions of the  $\sigma$ –models related by the Poisson–Lie T–plurality were given in [7]. The Poisson–Lie T–duality is a special case of Poisson–Lie T–plurality with  $K = S = 0$ ,  $Q = R = \mathbf{1}$ .

### 3 Poisson–Lie T–plurality transformation of renormalization group equation

It was shown in [2] that the one-loop renormalization group equations for Poisson–Lie dualizable  $\sigma$ -models

$$\frac{dM^{ab}}{dt} = r^{ab}, \quad r^{ab} = R^{as}{}_t L^{tb}{}_s \quad (9)$$

where

$$R^{ab}{}_c = \frac{1}{2}(M_S^{-1})_{cd} (A^{ab}{}_e M^{de} + B^{ad}{}_e M^{eb} - B^{db}{}_e M^{ae}), \quad (10)$$

$$L^{ab}{}_c = \frac{1}{2}(M_S^{-1})_{cd} (B^{ab}{}_e M^{ed} + A^{db}{}_e M^{ae} - A^{ad}{}_e M^{eb}), \quad (11)$$

$$A^{ab}{}_c = \tilde{f}^{ab}{}_c - f_{cd}{}^a M^{db}, \quad B^{ab}{}_c = \tilde{f}^{ab}{}_c + M^{ad} f_{dc}{}^b, \quad (12)$$

$$M_S = \frac{1}{2}(M + M^T) \quad (13)$$

are covariant with respect to the Poisson–Lie T–duality, namely that the equation (9) is equivalent to

$$\frac{d\tilde{M}^{ab}}{dt} = \tilde{r}^{ab}, \quad \tilde{r}^{ab} = \tilde{R}^{as}{}_t \tilde{L}^{tb}{}_s. \quad (14)$$

obtained by

$$f \leftrightarrow \tilde{f}, \quad M \leftrightarrow \tilde{M} = M^{-1}. \quad (15)$$

This equivalence can be reexpressed as

$$\tilde{r} = -M^{-1} \cdot r \cdot M^{-1}. \quad (16)$$

One may naturally expect that the equations (9) are covariant also with respect to the Poisson–Lie T–plurality when

$$f \rightarrow \hat{f}, \quad \tilde{f} \rightarrow \bar{f}, \quad M \rightarrow \hat{M}, \quad (17)$$

and  $\hat{M}$  is given by (7). The condition for equivalence of renormalization group equations now becomes a bit more complicated. It follows from the transformation properties (7) of the matrix  $M$  that

$$\frac{d\hat{M}}{dt} = (M \cdot Q + S)^{-1} \frac{dM}{dt} (K - Q \cdot \hat{M}). \quad (18)$$

Applying the renormalization group equations (9) for both Poisson–Lie T–plurality related models we get a necessary condition for the one–loop equivalence of the renormalization group equations

$$\hat{r} = (M \cdot Q + S)^{-1} \cdot r \cdot (K - Q \cdot \hat{M}). \quad (19)$$

This equation is satisfied for some decompositions of Drinfel’d doubles, described in 4.2.1, but not in general. The simplest case of Poisson–Lie T–plurality where (19) is not satisfied will be shown in 4.1.

On the other hand it was noted in [1] that there is a certain ambiguity in the one–loop renormalization group equations. Namely, the flow given by the equation (9) is physically equivalent to the one given by the equation

$$\frac{dM^{ab}}{dt} = r^{ab} + R^{ab}{}_c \xi^c, \quad (20)$$

where  $\xi^c$  are arbitrary constants.

The origin of these arbitrary constants  $\xi^c$  lies in the fact that the metric and B–field are determined up to the choice of coordinates, i.e. up to a diffeomorphism, of the group  $G$  viewed as a manifold. In our case we may in addition require that the transformed action takes again the form (1), (2) for some matrix  $M'$ . On the other hand, we do not have to require the diffeomorphism to be a group homomorphism because the group structure plays only an auxiliary role in the physical interpretation.

For example, in the particular case of semi–Abelian double, i.e.  $\Pi = 0$ , with a symmetric matrix  $M$ , the left translation by an arbitrary group element  $h = \exp(X) \in G$ , i.e. replacement of  $g$  by  $hg$  in the action (1), leads to the new matrix  $M' = Ad(h) \cdot M \cdot Ad(h)$ , specifying a metric physically equivalent to the original one. Such a diffeomorphism is generated by the flow of the left–invariant vector field  $X$ . For general Manin triples and matrices  $M$  similar transformations are generated by more complicated vector fields parameterized by the constants  $\xi^c$ , as was found in [1]. Thus the renormalization group flows (20) differing by the choice of  $\xi^c$  are physically equivalent if

$$f \leftrightarrow \tilde{f}, \quad M \leftrightarrow M^{-1}, \quad \xi^a \leftrightarrow \tilde{\xi}_a = -(M^{-1})_{ab} \xi^b. \quad (21)$$

One would expect that the covariance of the modified renormalization group equations (20) is obtained by fixing the value of  $\hat{\xi}_a$  once original constants  $\xi_a$  and the transformation to the plural model (4) are chosen, or, at the very

least, that the equivalence of the renormalization group flows

$$\hat{r}^{ab} + \hat{R}^{ab}{}_c \hat{\xi}^c = \left[ (M \cdot Q + S)^{-1} \cdot (r + R \cdot \xi) \cdot (K - Q \cdot \hat{M}) \right]^{ab} \quad (22)$$

under the plurality transformation (4) fixes both  $\xi_a$  and  $\hat{\xi}_a$ . Unfortunately, as we shall show in the section 4.2.2, the condition (22) is in general not satisfied for any choice of the constants  $\xi^c, \hat{\xi}^c$  although there are some intriguing exceptions mentioned before.

The constants  $\xi^c$  may also play another role in the renormalization group equations. The ambiguity in their choice can be in some cases partially or fully fixed by supplementary conditions like consistence of the renormalization group equations with a selected ansatz (truncation) for the matrix  $M$ , e.g.  $M$  diagonal.

## 4 Examples of quantum (non)equivalence under Poisson–Lie T–plurality

### 4.1 2-dimensional plurality

In the dimension four there is only one Drinfel’d double containing more than two (mutually dual) Manin triples [8]. This Drinfel’d double can be decomposed either into semi–Abelian  $(A|S)$  or type B non–Abelian Manin triple  $(S|S')$  (and their duals). Their nontrivial Lie brackets are

$$(A|S): \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1, \quad (23)$$

and  $(S|S')$ :

$$\begin{aligned} [\hat{T}_1, \hat{T}_2] &= \hat{T}_2, \quad [\bar{T}^1, \bar{T}^2] = \bar{T}^1, \\ [\hat{T}_1, \bar{T}^1] &= \hat{T}_2, \quad [\hat{T}_1, \bar{T}^2] = -\hat{T}_1 - \bar{T}^2, \quad [\hat{T}_2, \bar{T}^2] = \bar{T}^1. \end{aligned} \quad (24)$$

A transformation matrix  $(A|S) \rightarrow (S|S')$  is

$$\begin{pmatrix} K & Q \\ R & S \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

The background for the Poisson–Lie dualizable  $\sigma$ –model given by the Manin triple  $(A|S)$  and the matrix

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \quad (26)$$

is

$$E_{ab}(g) = \begin{pmatrix} \frac{M_2}{M_1 M_2 + x_2^2} & -\frac{x_2}{M_1 M_2 + x_2^2} \\ \frac{x_2}{M_1 M_2 + x_2^2} & \frac{M_1}{M_1 M_2 + x_2^2} \end{pmatrix}. \quad (27)$$

Background for the  $\sigma$ –model obtained by the Poisson–Lie–plurality transformation (25)

$$\hat{E}_{ab}(g) = \begin{pmatrix} \frac{M_1 M_2}{M_1(e^{x_1}-2)^2 + M_2(e^{x_1}-1)^2} & -\frac{M_1(e^{x_1}-2) + M_2(e^{x_1}-1)}{M_1(e^{x_1}-2)^2 + M_2(e^{x_1}-1)^2} \\ \frac{M_1(e^{x_1}-2) + M_2(e^{x_1}-1)}{M_1(e^{x_1}-2)^2 + M_2(e^{x_1}-1)^2} & \frac{1}{M_1(e^{x_1}-2)^2 + M_2(e^{x_1}-1)^2} \end{pmatrix} \quad (28)$$

is given by the Manin triple  $(S|S')$  and the matrix

$$\hat{M} = \begin{pmatrix} \frac{1}{M_1 + M_2} & -\frac{M_1}{M_1 + M_2} \\ \frac{M_1}{M_1 + M_2} & \frac{M_1 M_2}{M_1 + M_2} \end{pmatrix} \quad (29)$$

obtained by the formula (7).

The right–hand side of the renormalization group equation (9) for the  $\sigma$ –model with the background (27) is

$$r^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad (30)$$

so that

$$M'_1(t) = -1, \quad M'_2(t) = 0. \quad (31)$$

The right–hand side of the renormalization group equation for the plurality transformed  $\sigma$ –model is

$$\hat{r}^{ab} = \begin{pmatrix} \frac{1}{(M_1 + M_2)^2} & -\frac{2M_1 + M_2}{(M_1 + M_2)^2} \\ \frac{2M_1 + M_2}{(M_1 + M_2)^2} & -\frac{(2M_1 + M_2)^2}{(M_1 + M_2)^2} \end{pmatrix}. \quad (32)$$

Invariance of the renormalization group equations under Poisson–Lie T–plurality requires that if we insert (31) into  $d\hat{M}/dt$  we get  $\hat{r}$ .

$$\left. \frac{d\hat{M}}{dt} \right|_{M'(t)=r} = \hat{r}. \quad (33)$$

It is easy to check from (29),(31) and (32) that in this case the equation (33) is not satisfied. However, the invariance is restored when we employ the ambiguity in the definition of the r.h.s of the renormalization group equation

$$r^{ab} \rightarrow r^{ab} + R^{ab}{}_c \xi^c, \quad \hat{r}^{ab} \rightarrow \hat{r}^{ab} + \hat{R}^{ab}{}_c \hat{\xi}^c. \quad (34)$$

The conditions (22) yield

$$\xi_1 = \frac{4M_1}{M_2} + 1, \quad \xi_2 = -2, \quad \hat{\xi}_1 = -\frac{1}{M_1 + M_2}, \quad \hat{\xi}_2 = \frac{M_2}{M_1 + M_2} \quad (35)$$

and the covariance of the equations (34) is restored by this choice.

Let us note that the conditions (22) in this case represent 4 linear inhomogeneous equations for 4 unknowns so that it is not all that surprising that we get appropriate  $\xi$  and  $\hat{\xi}$ .

## 4.2 3-dimensional plurality

In the dimension six there are many Drinfel'd doubles containing more than two mutually dual Manin triples. Therefore they offer many possibilities to test the equivalence of the renormalization group equations under the Poisson–Lie T–plurality. We shall present examples of classically Poisson–Lie plural  $\sigma$ –models where the equations are and are not (one–loop) equivalent. In the latter case the equivalence cannot be achieved even by suitable choices of constants  $\xi^c, \hat{\xi}^c$  in the Eq. (22).

### 4.2.1 Drinfel'd double $(5|1) \rightarrow (6_0|1)$

First we shall consider the Drinfel'd double which can be decomposed into the Manin triple  $(5|1)$  or  $(6_0|1)$  [9]. Their nontrivial Lie brackets are:

$(5|1)$  :

$$\begin{aligned} [T_1, T_2] &= -T_2, \quad [T_3, T_1] = T_3, \quad [T_1, \tilde{T}^2] = \tilde{T}^2 \\ [T_1, \tilde{T}^3] &= \tilde{T}^2, \quad [T_2, \tilde{T}^2] = -\tilde{T}^1, \quad [T_3, \tilde{T}^3] = -\tilde{T}^1. \end{aligned}$$

$(6_0|1)$  :

$$\begin{aligned} [\hat{T}_2, \hat{T}_3] &= \hat{T}_1, \quad [\hat{T}_3, \hat{T}_1] = -\hat{T}_2, \quad [\hat{T}_1, \bar{T}^2] = -\bar{T}^2, \\ [\hat{T}_2, \bar{T}^1] &= -\bar{T}^2, \quad [\hat{T}_3, \bar{T}^1] = \bar{T}^2, \quad [\bar{T}_3, \hat{T}^2] = \bar{T}^1. \end{aligned}$$



A matrix of isomorphism between these two decompositions of the Drinfel'd double can be chosen as

$$(5|1) \rightarrow (6_0|1) : \quad \begin{pmatrix} K & Q \\ R & S \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \quad (36)$$

In this case the equation (19) is satisfied for a general matrix  $M$  and thus the renormalization group equations are equivalent. Besides that for any given constants  $\xi^c$  one finds unique constants  $\hat{\xi}^c$  such that the equation (22) holds. For a generic matrix  $M = (M_{ij})$  their values are

$$\begin{aligned} \hat{\xi}^1 &= \frac{(2M_{32} + 1)\xi^2 - 2M_{22}\xi^3}{2M_{22}}, & \hat{\xi}^2 &= \frac{2M_{22}\xi^3 + (1 - 2M_{32})\xi^2}{2M_{22}}, \\ \hat{\xi}^3 &= \frac{M_{12}\xi^2 - M_{22}\xi^1}{M_{22}}. \end{aligned} \quad (37)$$

#### 4.2.2 Manin triples (4|1) and (4|2.i)

An example of Poisson–Lie  $\sigma$ -models where the plurality is not preserved by quantization is provided by  $\sigma$ -models obtained from the Manin triples (4|1) and (4|2.i) that are decompositions of the same Drinfel'd double [9]. Their nontrivial Lie brackets are

(4|1) :

$$\begin{aligned} [T_1, T_2] &= -T_2 + T_3, \quad [T_3, T_1] = T_3, \\ [T_1, \tilde{T}^2] &= \tilde{T}^2, \quad [T_1, \tilde{T}^3] = -\tilde{T}^2 + \tilde{T}^3, \\ [T_2, \tilde{T}^2] &= -\tilde{T}^1, \quad [T_2, \tilde{T}^3] = \tilde{T}^1, \quad [T_3, \tilde{T}^3] = -\tilde{T}^1. \end{aligned}$$

(4|2.i) :

$$\begin{aligned} [\hat{T}_1, \hat{T}_2] &= -\hat{T}_2 + \hat{T}_3, \quad [\hat{T}_3, \hat{T}_1] = \hat{T}_3, \\ [\hat{T}_1, \bar{T}^2] &= \hat{T}_3 + \bar{T}^2, \quad [\hat{T}_1, \bar{T}^3] = -\hat{T}_2 - \bar{T}^2 + \bar{T}^3, \\ [\hat{T}_2, \bar{T}^2] &= -\bar{T}^1, \quad [\hat{T}_2, \bar{T}^3] = \bar{T}^1, \quad [\hat{T}_3, \bar{T}^3] = -\bar{T}^1, \\ [\bar{T}^2, \bar{T}^3] &= \bar{T}^1. \end{aligned}$$

Transformation matrix between bases of  $(4|1)$  and  $(4|2.i)$  is

$$\begin{pmatrix} K & Q \\ R & S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix} \quad (38)$$

Background for the  $\sigma$ -model obtained from the Manin triple  $(4|1)$  and matrix  $M$  is given by  $E(g) = E_0 = M^{-1}$ . The renormalization group equation for this  $\sigma$ -model obtained from the Eq. (20) when the matrix  $M$  is diagonal is

$$\frac{d}{dt} \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} = \begin{pmatrix} \frac{M_1^2(M_2+4M_3)}{2M_3} & 0 & 0 \\ 0 & \frac{M_1M_2^2}{2M_3} & 0 \\ 0 & 0 & -\frac{M_1M_2}{2} \end{pmatrix}. \quad (39)$$

Background for the Poisson-Lie plurality transformed  $\sigma$ -model given by the Manin triple  $(4|2.i)$  and matrix

$$\hat{M} = M + R = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & \frac{1}{2} \\ 0 & -\frac{1}{2} & M_3 \end{pmatrix} \quad (40)$$

obtained from the formula (7) and (38) is given by the bilinear form

$$\hat{E}(\hat{g}) = \begin{pmatrix} \frac{1}{M_1} & 0 & 0 \\ 0 & \frac{4M_3e^{4x_1}}{4M_2M_3e^{4x_1}+(1-2e^{2x_1})^2} & \frac{2e^{2x_1}-4e^{4x_1}}{4M_2M_3e^{4x_1}+(1-2e^{2x_1})^2} \\ 0 & \frac{4e^{4x_1}-2e^{2x_1}}{4M_2M_3e^{4x_1}+(1-2e^{2x_1})^2} & \frac{4M_2e^{4x_1}}{4M_2M_3e^{4x_1}+(1-2e^{2x_1})^2} \end{pmatrix} \quad (41)$$

The right-hand side of the renormalization group equation for the plurality transformed  $\sigma$ -model is

$$\hat{r}^{ab} = \begin{pmatrix} \frac{M_1^2(M_2^2+4M_2M_3+4)}{2M_2M_3} & 0 & 0 \\ 0 & \frac{M_1(M_2^2-4)}{2M_3} & 2M_1 \\ 0 & -2M_1 & -\frac{M_1(M_2^2+4)}{2M_2} \end{pmatrix}. \quad (42)$$

and similarly as in the 2-dimensional case, the condition (19) of covariance of the renormalization group equations under Poisson–Lie T-plurality (33) is not satisfied.

If we employ the ambiguity in the choice of  $\xi^c, \hat{\xi}^c$  on the r.h.s. of the renormalization group equations then we get

$$r^{ab} + R^{ab}{}_c \xi^c = \begin{pmatrix} \frac{M_1^2(M_2+4M_3)}{2M_3} & \frac{1}{2}M_1 \left( \frac{M_2\xi_3}{M_3} - 2\xi_2 \right) & \frac{1}{2}M_1(\xi_2 - 2\xi_3) \\ -\frac{M_1M_2\xi_3}{2M_3} & \frac{M_1M_2^2}{2M_3} + \xi_1M_2 & -\frac{1}{2}M_2\xi_1 \\ \frac{1}{2}M_1\xi_2 & -\frac{1}{2}M_2\xi_1 & M_3\xi_1 - \frac{M_1M_2}{2} \end{pmatrix} \quad (43)$$

and

$$\hat{r}^{ab} + \hat{R}^{ab}{}_c \hat{\xi}^c = \begin{pmatrix} \frac{M_1^2(M_2^2+4M_2M_3+4)}{2M_2M_3} & \frac{M_1(M_2\hat{\xi}^3-2M_3\hat{\xi}^2+2\xi^3)}{2M_3} & \frac{M_1(M_2(\hat{\xi}^2-2\xi^3)-2\xi^2)}{2M_2} \\ -\frac{M_1(M_2-2)\hat{\xi}^3}{2M_3} & \frac{M_1(M_2^2-4)}{2M_3} + M_2\hat{\xi}^1 & 2M_1 - \frac{1}{2}M_2\hat{\xi}^1 + \hat{\xi}^1 \\ \frac{M_1(M_2-2)\hat{\xi}^2}{2M_2} & -2M_1 - \frac{1}{2}M_2\hat{\xi}^1 - \hat{\xi}^1 & M_3\hat{\xi}^1 - \frac{M_1(M_2^2+4)}{2M_2} \end{pmatrix}. \quad (44)$$

The condition (22) now represents 9 linear inhomogeneous equations for 6 unknowns  $\xi^c, \hat{\xi}^c$  which have no solution. Thus the renormalization group equations are not covariant under Poisson–Lie T-plurality in this case.

### 4.2.3 The other Manin triples

We have checked numerous examples of plurality based on transformation matrices in (4) and have found only one additional example, namely  $(4|1) \cong (6_0|2)$ , such that one-loop equivalence of renormalization group flows (16) holds under plurality for a general matrix  $M$ .

### 4.2.4 Relation to the renormalization group equations on the Drinfel'd double

The above presented results appear rather surprising when compared to those of [3]. In that paper the renormalization group equations

$$\frac{dR_{AB}}{dt} = S_{AB}(R, h) = \frac{1}{4}(R_{AC}R_{BF} - \eta_{AC}\eta_{BF})(R^{KD}R^{HE} - \eta^{KD}\eta^{HE})h_{KH}^C h_{DE}^F \quad (45)$$

on the whole Drinfel'd double for the symmetric matrix  $R$  were derived. For a given decomposition of the Drinfel'd double into a Manin triple  $(G|\tilde{G})$ ,

the structure constants  $h$  of the Drinfel'd double are given by the structure constants  $f, \tilde{f}$  of the subalgebras of the Manin triple  $h = h(f, \tilde{f})$  as in equation (5). The matrix  $R$  is related to the matrix  $M$ , which defines the  $\sigma$ -model on the group  $G$ , by

$$R_{AB} = \rho_{AB}(M) = \begin{pmatrix} \tilde{M}_s - B\tilde{M}_s^{-1}B & -B\tilde{M}_s^{-1} \\ \tilde{M}_s^{-1}B & \tilde{M}_s^{-1} \end{pmatrix}, \quad (46)$$

where

$$B = \frac{1}{2} [M^{-1} - (M^{-1})^t], \quad \tilde{M}_s = \frac{1}{2} [M^{-1} + (M^{-1})^t],$$

$$R^{AB} = (R^{-1})^{AB}, \quad R^{-1} = \eta \cdot R \cdot \eta,$$

and

$$\eta_{AB} = \langle T_A | T_B \rangle = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{d_g \times d_g} \\ \mathbb{I}_{d_g \times d_g} & \mathbf{0} \end{pmatrix}, \quad (47)$$

where  $T_A = \{T_i, \tilde{T}^j\}$ .

A general proof that the renormalization group equations (45) on the Drinfel'd double are equivalent to those on the group  $G$ , namely to the equations (9), was not given<sup>1</sup> in [3]. Therefore we tried to check it independently on examples. It is easy to show that the equivalence of (45) and (9) where  $r^{ab} = r^{ab}(M, f, \tilde{f})$  requires

$$S_{AB} \left( \rho(M), h(f, \tilde{f}) \right) = \frac{\partial \rho_{AB}}{\partial M^{ab}}(M) r^{ab}(M, f, \tilde{f}). \quad (48)$$

Unfortunately, for nonsymmetric matrices  $M$  and Manin triples that are not semi-Abelian this equation does not hold in general.

The simplest example of that are renormalization group equations for the Manin triple  $(S|S')$  where the structure constants  $\hat{f}, \hat{\bar{f}}$  follow from the commutation relations (24) and the matrix  $\hat{M}$  given by (29) is of the form

$$\hat{M} = \begin{pmatrix} \hat{M}_1 & -\hat{M}_2 \\ \hat{M}_2 & \hat{M}_3 \end{pmatrix}. \quad (49)$$

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<sup>1</sup>A sketch of it was presented in the thesis [10] but many tedious computational details were omitted in it.

We have

$$\rho(\hat{M}) = \begin{pmatrix} \frac{1}{\hat{M}_1} & 0 & 0 & -\frac{\hat{M}_2}{\hat{M}_1} \\ 0 & \frac{1}{\hat{M}_3} & \frac{\hat{M}_2}{\hat{M}_3} & 0 \\ 0 & \frac{\hat{M}_2}{\hat{M}_3} & \frac{\hat{M}_2^2}{\hat{M}_3} + \hat{M}_1 & 0 \\ -\frac{\hat{M}_2}{\hat{M}_1} & 0 & 0 & \frac{\hat{M}_2^2}{\hat{M}_1} + \hat{M}_3 \end{pmatrix}, \quad (50)$$

$$S_{AB} \left( \rho(\hat{M}), h(\hat{f}, \bar{f}) \right) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & \frac{(\hat{M}_2-1)^2}{\hat{M}_3^2} & \frac{(\hat{M}_2-1)(\hat{M}_2^2-\hat{M}_2+\hat{M}_1\hat{M}_3)}{\hat{M}_3^2} & 0 \\ 0 & \frac{(\hat{M}_2-1)(\hat{M}_2^2-\hat{M}_2+\hat{M}_1\hat{M}_3)}{\hat{M}_3^2} & \frac{(\hat{M}_2^2-\hat{M}_2+\hat{M}_1\hat{M}_3)^2}{\hat{M}_3^2} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

but

$$\frac{\partial \rho_{AB}}{\partial \hat{M}^{ab}} r^{ab}(\hat{M}, \hat{f}, \bar{f}) = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & \frac{(\hat{M}_2+1)^2}{\hat{M}_3^2} & \frac{(\hat{M}_2+1)(\hat{M}_2^2+\hat{M}_2+\hat{M}_1\hat{M}_3)}{\hat{M}_3^2} & 0 \\ 0 & \frac{(\hat{M}_2+1)(\hat{M}_2^2+\hat{M}_2+\hat{M}_1\hat{M}_3)}{\hat{M}_3^2} & \frac{(\hat{M}_2^2+\hat{M}_2+\hat{M}_1\hat{M}_3)^2}{\hat{M}_3^2} & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

## 5 Renormalizable $\sigma$ -models for $M$ proportional to the unit or diagonal matrix

The simplest ansatz for the constant matrix is  $M = m\mathbf{1}$  where  $\mathbf{1}$  is the identity matrix and  $m \neq 0$ . As mentioned in the Introduction, truncation or symmetry of the constant matrix  $M$  that determines the background of the  $\sigma$ -model often contradicts the form of the r.h.s of the renormalization group equations (9). On the other hand, the freedom in the choice of  $\xi^c$  in (20) may help to restore the renormalizability. It is therefore of interest to find consistency conditions for the renormalization group equations for the  $\sigma$ -models given by this simple  $M$ .

Two-dimensional Poisson-Lie  $\sigma$ -models are given by Manin triples generated by abelian or solvable Lie algebras with Lie products

$$[T_1, T_2] = a T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{a} \tilde{T}^2, \quad a \in \{0, 1\}, \quad \tilde{a} \in \mathbf{R} \quad (51)$$

or

$$[T_1, T_2] = T_2, \quad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1 \quad (52)$$

In the former case, the equation (20) for  $M = m\mathbf{1}$  reads

$$\begin{pmatrix} \frac{dm}{dt} & 0 \\ 0 & \frac{dm}{dt} \end{pmatrix} = \begin{pmatrix} a^2 m^2 - \tilde{a}^2 & (a m + \tilde{a})\xi^2 \\ 0 & -(a m + \tilde{a})\xi^1 \end{pmatrix} \quad (53)$$

so that we generically get  $\xi^1 = \tilde{a} - a m$ ,  $\xi^2 = 0$  and the renormalization group equation is  $dm/dt = a^2 m^2 - \tilde{a}^2$ . In the special case  $a = 1$ ,  $m = -\tilde{a}$  the r.h.s. of the equation (53) vanishes for all choices of  $\xi^k$ , i.e. there is no renormalization. Notice that had we allowed a diagonal ansatz

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (54)$$

instead of the multiple of the unit matrix, the restriction on the value of  $\xi^1$  would disappear and the renormalization group equation would take the form

$$\frac{dm_1}{dt} = -\tilde{a}^2 + m_1^2 a^2, \quad \frac{dm_2}{dt} = -\frac{m_2}{m_1} \xi^1 (\tilde{a} + m_1 a). \quad (55)$$

For the Manin triple (52), the equation (20) reads

$$\begin{pmatrix} \frac{dm}{dt} & 0 \\ 0 & \frac{dm}{dt} \end{pmatrix} = \begin{pmatrix} m^2 + \xi^2 & m(\xi^2 - 1) \\ m - \xi^1 & -1 - m\xi^1 \end{pmatrix} \quad (56)$$

and no choice of  $\xi^1, \xi^2$  satisfies the equation (56). Therefore the Poisson–Lie  $\sigma$ –model given by Manin triple (52) is not renormalizable with  $M$  kept proportional to the unit matrix. The situation changes when we allow general diagonal form (54) of the matrix  $M$ . Then the renormalization group equation becomes

$$\begin{pmatrix} \frac{dm_1}{dt} & 0 \\ 0 & \frac{dm_2}{dt} \end{pmatrix} = \begin{pmatrix} m_1^2 + \frac{m_1}{m_2} \xi^2 & m_1(\xi^2 - 1) \\ m_1 - \xi^1 & -1 - m_2 \xi^1 \end{pmatrix} \quad (57)$$

which allows the flow

$$\frac{dm_1}{dt} = m_1^2 + \frac{m_1}{m_2}, \quad \frac{dm_2}{dt} = -1 - m_1 m_2$$

respecting the diagonal ansatz (54) for the unique choice of the constants  $\xi^1 = m_1$ ,  $\xi^2 = 1$ .

Solvability of one-loop renormalization group equations for three-dimensional Poisson–Lie  $\sigma$ -models with  $M$  proportional to the unit matrix fixes the constant  $\xi^3 = 0$  and is consistent with the choice  $\xi^2 = 0$  (unique in some cases). It exists for the following Manin triples and choices of  $\xi^1$  and/or  $m$

$$(1|1) : \quad \frac{dm}{dt} = 0, \quad \xi^1 = 0, \quad (58)$$

$$(3|3.i|b) : \quad \frac{dm}{dt} = 0, \quad \xi^1 = 0, \quad m = \pm b, \quad (59)$$

$$(5|1) : \quad \frac{dm}{dt} = 2m^2, \quad \xi^1 = 2m, \quad (60)$$

$$(6_0|5.iii|b) : \quad \frac{dm}{dt} = 0, \quad \xi_1 = 0, \quad m = \pm b, \quad (61)$$

$$(6_a|6_{1/a}.i|b) : \quad \frac{dm}{dt} = 0, \quad \xi_1 = 0, \quad m = \pm b/a, \quad (62)$$

$$(6_a|6_{1/a}.i|b) : \quad \frac{dm}{dt} = 2b^2(a^2 - \frac{1}{a^2}), \quad \xi_1 = -2b(a + \frac{1}{a}), \quad m = -b, \quad (63)$$

$$(7_a|1) : \quad \frac{dm}{dt} = 2a^2m^2, \quad \xi^1 = 2am, \quad a \geq 0 \quad (64)$$

$$(7_a|7_{1/a}|b) : \quad \frac{dm}{dt} = 2(m^2 - b^2), \quad \xi^1 = 2(m - b), \quad a = 1, \quad (65)$$

$$(9|1) : \quad \frac{dm}{dt} = -m^2/2, \quad \xi^1 = 0, \quad (66)$$

$$(9|5|b) : \quad \frac{dm}{dt} = -\frac{1}{2}m^2 - 2b^2, \quad \xi^1 = -2b \quad (67)$$

and their duals (for notation of  $(X|Y)$  or  $(X|Y|b)$  see [9]). Renormalization of Poisson–Lie  $\sigma$ -model given by other six-dimensional Manin triples is not consistent with the assumption  $M$  proportional to identity, i.e. renormalization spoils the ansatz.

We have also investigated three-dimensional  $\sigma$ -models with general diagonal matrices  $M$  but the list of renormalizable models is rather long so that we do not display it here.

We notice that the list of renormalizable three-dimensional Poisson–Lie  $\sigma$ -models with  $M$  proportional to the unit matrix is in agreement with the results obtained in [11]. There the conformally invariant Poisson–Lie  $\sigma$ -models, i.e. those with vanishing  $\beta$ -function, were studied and the sigma models with diagonal  $M$  and constant dilaton field were obtained. They appear in the above constructed list with vanishing r.h.s of the renormalization

group equation.

## 6 Conclusions

We have discussed the transformation properties of the renormalization group flow under Poisson–Lie T–plurality.

Originally we expected on the basis of our previous experience with the Poisson–Lie T–duality and T–plurality that it is possible to generalize the proof of the equivalence of the renormalization group flows of Poisson–Lie T–dual sigma models [2] to the case of Poisson–Lie T–plurality although we supposed that there will be technical difficulties arising from the relative complexity of the transformation formula (7) compared to the duality case (15). Instead we have found that the properties of Poisson–Lie T–plurality significantly differ from Poisson–Lie T–duality in this aspect, i.e. that the renormalization group flows in the former case are in general not equivalent. On the other hand we have found some exceptions to this rule whose significance is not yet clear.

Next, we studied whether the situation is changed when the freedom in the choice of constants  $\xi^c$  in the renormalization group equations (22) is employed. We have found that it does not help with the exception of the 2–dimensional targets where the equivalence can be reintroduced by a suitable choice of  $\xi^c, \hat{\xi}^c$  on dimensional grounds.

Comparing our results to those of [3] we see that there is an incompatibility between the renormalization at the level of the Drinfel’d double [3], i.e. the equation (45), and at the level of the individual Poisson–Lie subgroups [2], i.e. the equation (9). What is the reason for this discrepancy and how it can be resolved is not yet clear. We are inclined to believe that the correct renormalization group equations are given by the covariant formulation (45) derived in [3] and that the equations (9) on the subgroups of the Drinfel’d double need to be somehow modified.

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